On $b$-coloring of cartesian product of graphs

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Abstract

A $b$-coloring of a graph $G$ by $k$ colors is a proper $k$-coloring of the vertices of $G$ such that in each color class there exists a vertex having neighbors in all the other $k-1$ color classes. The $b$-chromatic number $\varphi(G)$ of a graph $G$ is the maximum $k$ for which $G$ has a $b$-coloring by $k$ colors. This concept was introduced by R.W. Irving and D.F. Manlove in 1999. In this paper we study the $b$-chromatic numbers of the cartesian products of paths and cycles with complete graphs and the cartesian product of two complete graphs.

Key Words: $b$-chromatic number, $b$-coloring, dominating coloring.

1 Introduction

Let $G$ be a graph without loops and multiple edges with vertex set $V(G)$ and edge set $E(G)$. A proper $k$-coloring of graph $G$ is a function $c$ defined on the $V(G)$, onto a set of colors $C = \{1, 2, \ldots, k\}$ such that any two adjacent vertices have different colors. In fact, for every $i$, $1 \leq i \leq k$, the set $c^{-1}(\{i\})$ is an independent set of vertices which is called a color class. The minimum cardinality $k$ for which $G$ has a proper $k$-coloring is the chromatic number $\chi(G)$ of $G$.

A $b$-coloring of a graph $G$ by $k$ colors is a proper $k$-coloring of the vertices of $G$ such that in each color class $i$ there exists a vertex $x_i$ having neighbors in all the other $k-1$ color classes. We will call such a vertex $x_i$, a $b$-dominating vertex and the set of vertices $\{x_1, x_2, \ldots, x_k\}$ a $b$-dominating system. The $b$-chromatic number $\varphi(G)$ of a graph $G$ is the maximum $k$ for which $G$ has a $b$-coloring by
The \( b \)-chromatic number was introduced by R.W. Irving and D.F. Manlove in [2]. They proved that determining \( \varphi(G) \) is NP-hard for general cases, but it is polynomial for trees. An immediate and useful bounds for \( \varphi(G) \) is:

\[
\chi(G) \leq \varphi(G) \leq \Delta(G) + 1,
\]

where \( \Delta(G) \) is the maximum degree of vertices in \( G \).

The cartesian product of two graphs \( G_1 \) and \( G_2 \), denoted by \( G_1 \square G_2 \), is a simple graph with \( V(G_1) \times V(G_2) \) as its vertex set and two vertices \((u_1, v_1)\) and \((u_2, v_2)\) are adjacent in \( G_1 \square G_2 \) if and only if either \( u_1 = u_2 \) and \( v_1, v_2 \) are adjacent in \( G_2 \), or \( u_1, u_2 \) are adjacent in \( G_1 \) and \( v_1 = v_2 \). In the sequel, where \( |V(G_1)| = m \) and \( |V(G_2)| = n \), we consider the vertex set of the graph \( G_1 \square G_2 \), as an \( m \times n \) array in which the entry \((i, j)\) corresponds to the vertex \((i, j), i \in V(G_1) \) and \( j \in V(G_2) \), and each column induces a copy of graph \( G_1 \) and each row induces a copy of graph \( G_2 \). In Section 3, where \( G_2 = C_n \), the neighbors of entry \((i, j)\) in the row \( i \) are entries \((i, j \pm 1)\). In Section 4, where \( G_2 = P_n \), the neighbors of entry \((i, j)\) in the row \( i \) are entries \((i, j \pm 1)\), for \( 2 \leq j \leq n - 2 \) and for \( j = 1 \) and \( j = n \) are \((i, 2)\) and \((i, j - 1)\), respectively. So through this paper all first components of entries are modulo \( |V(G_1)| = m \) and all second components of entries are modulo \( |V(G_2)| = n \).

The \( b \)-chromatic number of the cartesian product of some graphs such as \( K_{1,n} \square K_{1,n} \), \( K_{1,n} \square P_k \), \( P_n \square P_k \), \( C_n \square C_k \) and \( C_n \square P_k \) was studied in [3]. In this paper we study the \( b \)-chromatic numbers of the cartesian products of paths and cycles with complete graphs and the cartesian product of two complete graphs.

## 2 \( b \)-chromatic number of graph \( K_m \square G \)

In this section we present some results on the \( b \)-chromatic number of the cartesian product of the complete graphs with every graph \( G \).

**Proposition 1.** Let \( c \) be a \( b \)-coloring of graph \( K_m \square G \) by \( \varphi \) colors, where \( \varphi > m \), and \( v \in V(G) \). Then the column corresponding to the vertex \( v \), contains at most \( \deg_G(v) \) \( b \)-dominating vertices.
Proof. By assumption $\varphi > m$, therefore in the $b$-coloring $c$ there is at least one color that does not appear in the column corresponding to the vertex $v$ of $G$, we denote this column by $K_m^v$. On the other hand this missing color must appear in the neighbors of all $b$-dominating vertices in $K_m^v$, which are obviously in different columns. Therefore the number of $b$-dominating vertices in $K_m^v$ is at most $\deg_G(v)$.

If $d = (d_1, d_2, \ldots, d_n)$ is the degree sequence of a graph $G$ with $n$ vertices, then by Proposition 1, in graph $K_m \circ G$ each column, denoted by $K_m^{(i)}$, $1 \leq i \leq n$, contains at most $d_i$ $b$-dominating vertices. Therefore, every $b$-dominating system of $G$ contains at most $\sum_{i=1}^n d_i$ vertices. So we have the following upper bounds for $\varphi(K_m \circ G)$ which improves the given upper bounds in [3].

**Corollary 1.** If $d = (d_1, d_2, \ldots, d_n)$ is the degree sequence of graph $G$ with $n$ vertices and $e$ edges, then

$$\varphi(K_m \circ G) \leq \sum_{i=1}^n d_i = 2e.$$  

Now we prove a lemma on completing a partial proper coloring of graph $K_m \circ G$ for every graph $G$. A partial proper coloring of a graph is an assignment of colors to some vertices of $G$, such that the adjacent vertices receive different colors.

Let $S_1, \ldots, S_n$ be some sets. A system of distinct representatives (SDR) for these sets is an $n$-tuple $(x_1, \ldots, x_n)$ of elements with the properties that $x_i \in S_i$ for $i = 1, \ldots, n$ and $x_i \neq x_j$ for $i \neq j$. It is a well known theorem that the family of sets $S_i$ has an SDR if and only if it satisfies the Hall’s condition, which is for every subset $I \subseteq \{1, 2, \ldots, n\}$, $|\bigcup_{i \in I} S_i| \geq |I|$, [1].

**Lemma 1.** Let $G$ be a graph and $m$ be a positive integer, which $m \geq 2\Delta(G)$. If $c$ is a partial proper coloring of graph $K_m \circ G$ by $m$ colors, such that each column has no uncolored vertices or at least $2\Delta(G)$ uncolored vertices, then $c$ can be extended to a proper coloring of graph $K_m \circ G$ by $m$ colors.

**Proof.** In a partial proper coloring of graph $K_m \circ G$ by $m$ colors, consider a column with $k \geq 1$ uncolored vertices $v_1, v_2, \ldots, v_k$, where by assumption $k \geq$
2Δ(G). Without loss of generality we denote k missing colors by 1, 2, . . . , k. For each i = 1, 2, . . ., k, let Si be the set of colors that can be used to color the vertex vi, properly, so Si ⊆ {1, 2, . . ., k}. For extending this coloring to a proper coloring of this column, it is enough to find an SDR for the family of sets Si, 1 ≤ i ≤ k. For this purpose we show that the family of sets Si, 1 ≤ i ≤ k, satisfies the Hall’s condition. Let I ⊆ {1, 2, . . ., k}, which |I| = r.

If r ≤ Δ(G), then for some i0 ∈ I we have

|∪i∈I Si| ≥ |Si0| ≥ k − Δ(G) ≥ Δ(G) ≥ r = |I|.

If r > Δ(G), then ∪i∈I Si = {1, 2, . . ., k}. Because if a color say i0, 1 ≤ i0 ≤ k, does not appear in any set Si, i ∈ I, then each vertex vi, i ∈ I, has a neighbor say ui of color i0 in the row containing vi. Since all of the vertices ui have the same color, they are in different columns. Hence we must have r = |I| ≤ Δ(G), which is a contradiction. Therefore

|∪i∈I Si| = k ≥ |I|.

So the coloring of each column can be extended and the proof is completed. □

Proposition 2. For every two graphs G and H, if graph H’ is obtained by replacing one of the edges of H with a path of length 3, then φ(G□H’) ≥ φ(G□H).

Proof. Let e = xy be an edge in H and H’ be obtained by replacing e with the path xwzy. Moreover, assume that e is a b-coloring of graph G□H by φ(G□H) colors. We define a b-coloring c’ of graph G□H’ as follows. We color the vertices in the columns corresponding to the vertices w and z in H’ the same as the color of vertices in the columns y and x in the coloring c, respectively. Finally we color the rest of the vertices the same as the coloring c. It is easy to see that c’ is a proper coloring and the b-dominating system in c is a b-dominating system in c’.

□

Corollary 2. For every positive integers m, n,

φ(Km□Cn+2) ≥ φ(Km□Cn) and φ(Km□Pn+2) ≥ φ(Km□Cn).
Proof. Let \( \varphi(K_m \Box C_n) = k \). The graph \( C_{n+2} \) is obtained by replacing one edge \( e = xy \) in \( C_n \) by the path \( xwzy \). So by Proposition 2, there is a \( b \)-coloring \( c \) of graph \( K_m \Box C_{n+2} \) by \( k \) colors. Furthermore by the proof of Proposition 2, we see that there is no \( b \)-dominating vertex in the columns corresponding to the vertices \( w \) and \( z \) in the coloring \( c \). Thus \( c \) is also a \( b \)-coloring of graph \( K_m \Box P_{n+2} \), where \( P_{n+2} \) is obtained by deleting the edge \( wz \) in \( C_{n+2} \).

3 \( b \)-chromatic number of graph \( K_m \Box C_n \)

In this section we determine the exact value of \( \varphi(K_m \Box C_n) \). We know that \( \chi(K_m \Box C_n) = m \) and \( \Delta(K_m \Box C_n) = m + 1 \). Therefore by (1),

\[
m \leq \varphi(K_m \Box C_n) \leq m + 2.
\]

To prove our main theorem in this section, we need the following lemma.

Lemma 2. If \( c \) is a \( b \)-coloring of graph \( K_m \Box C_n \) by \( k \) colors and \( S \) is a \( b \)-dominating system in \( c \), such that:

(i) there is one \( b \)-dominating vertex, say \((r, s)\), \( r \neq m \), in a color class \( x \), such that the vertices \((r, s)\) and \((r, s \pm 1)\) are not in \( S \),

(ii) row \( m \) have no vertex in \( S \),

(iii) when \( n \) is odd, \( c(m, s - 1) \neq x \).

Then \( \varphi(K_{m+1} \Box C_n) \geq k + 1 \).

Proof. Without loss of generality we assume that \((r, s) = (1, 1)\). We present a \( b \)-coloring \( c' \) of graph \( K_{m+1} \Box C_n \) by \( k + 1 \) colors as follows:

\[
c'(i, j) = \begin{cases} 
  x & \text{if } (i, j) = (m + 1, 1), \\
  k + 1 & \text{if } (i, j) = (1, 1), \\
  k + 1 & \text{if } (i, j) = (m + 1, 2t), \ 1 \leq t \leq \lfloor \frac{n}{2} \rfloor, \\
  c(m, 2t - 1) & \text{if } (i, j) = (m + 1, 2t - 1), \ 2 \leq t \leq \lceil \frac{n}{2} \rceil, \\
  k + 1 & \text{if } (i, j) = (m, 2t - 1), \ 2 \leq t \leq \lceil \frac{n}{2} \rceil, \\
  c(i, j) & \text{otherwise.}
\end{cases}
\]

From the definition of \( c' \) and the property (iii) it is easy to see that \( c' \) is a proper coloring. Moreover, because of the properties (i), (ii) and since in coloring \( c' \) each
column has a vertex with color \( k + 1 \), every vertex in \( S \) is a \( b \)-dominating vertex in \( c' \). Also the vertex \((1, 1)\) is a \( b \)-dominating vertex with color \( k + 1 \). Therefore \( c' \) is a \( b \)-dominating coloring by \( k + 1 \) colors. \( \square \)

**Theorem 1.** For positive integers \( m, n \geq 4 \):

\[
\varphi(K_m \square C_n) = \begin{cases} 
m & \text{if } m \geq 2n, 
m + 1 & \text{if } m = 2n - 1, 
m + 2 & \text{if } m \leq 2n - 2. 
\end{cases}
\]

**Proof.** Assume \( m \geq 2n \). By Corollary 1, \( \varphi(K_m \square C_n) \leq 2n \). Hence by (2), we have \( \varphi(K_m \square C_n) = m \).

Now let \( m = 2n - 1 \), by Corollary 1, \( \varphi(K_m \square C_n) \leq 2n = m + 1 \). To prove the equality we present a \( b \)-coloring of graph \( K_m \square C_n \) by \( m + 1 \) colors.

Consider an \((m + 1) \times n\) array and fill some of the entries of this array as follows. We denote this partial proper coloring by \( c \). All second components of entries are modulo \( n \), \( 1 \leq j \leq n \), \( 1 \leq k \leq \lfloor \frac{n}{2} \rfloor \) and \( r = 0, 1 \).

\[
\begin{align*}
c(2\lceil \frac{j}{2} \rceil - r, j) &= 2j - r, 
c(2k, 2k - 2) &= 4k - 1, 
c(2k, 2k + 1) &= 4k - 3, 
c(m + 1, 2k - r) &= 4k + 2r - 3.
\end{align*}
\]

If \( n \) is odd, then we also define

\[
c(m + 1, n) = c(n, n - 1) = c(n + 1, 1) = 4.
\]

In Figure 1, this array with the filled entries for \( n = 4 \) is shown.

It is not hard to see that, this array with some filled entries is a partial proper coloring of graph \( K_{m+1} \square C_n \), which each column has three filled entries. Since \( m = 2n - 1 \geq 7 \), every column has at least 4 uncolored vertices. Hence by Lemma 1, \( c \) can be extended to a proper coloring of graph \( K_{m+1} \square C_n \) by \( m + 1 \) colors. Now to obtain the desired coloring, we delete the last row. Note that in this coloring of graph \( K_m \square C_n \), each column has exactly one missing color. The set of vertices \( \{ (2\lceil j/2 \rceil - r, j) \mid 1 \leq j \leq n, r = 0, 1 \} \) is a \( b \)-dominating system.
Because for $1 \leq k \leq \lfloor n/2 \rfloor$, the missing color of column $2k$ is $4k - 3$ which is the color of vertices $(2k, 2k + 1)$ and $(2k - 1, 2k - 1)$ and the missing color of column $2k - 1$ is $4k - 1$ which is the color of vertices $(2k, 2k - 2)$ and $(2k - 1, 2k)$.

Now assume $9 \leq m \leq 2n - 2$; by (2), $\varphi(K_m \sq C_n) \leq m + 2$. To show the equality, we present a $b$-coloring of graph $K_m \sq C_n$ by $m + 2$ colors. Consider an $(m + 2) \times n$ array and fill some of the entries of this array as follows. We denote this partial proper coloring by $c$. All second components of entries are modulo $n$ and the values are modulo $m + 2$, $1 \leq j \leq \lceil m/2 \rceil + 1$, $1 \leq k \leq \lfloor m/4 \rfloor$ and $r = 0, 1$.

$$c(2j - r, j) = 2j - r,$$
$$c(2k - r, 2k - 2) = 4k + r - 1, \quad c(2k - r, 2k + 1) = 4k + r - 3,$$
$$c(m + 1, 2k - r) = 4k + 2r - 3, \quad c(m + 2, 2k - r) = 4k + 2r - 2.$$

If $m \equiv 0, 3 \pmod{4}$, then we also define

$$c(\lceil m/2 \rceil + 2 - r, \lfloor m/2 \rfloor) = 6 - r,$$
$$c(\lceil m/2 \rceil + 2 - r, \lfloor m/2 \rfloor + 1) = 5 + r,$$
$$c(m + 1 + r, \lceil m/2 \rceil + 1) = 6 - r.$$

In Figure 2, this array with the filled entries for $m = 9$ and $n = 6$ is shown.

It is not hard to see that, this array with some filled entries is a partial proper coloring of graph $K_{m+2} \sq C_n$, which each column has four filled entries. Since $m \geq 9$, every column has at least 4 uncolored vertices. Hence by Lemma 1, $c$ can be extended to a proper coloring of graph $K_{m+2} \sq C_n$ by $m + 2$ colors. Now to
obtain the desired coloring, we delete the last two rows. Note that in this coloring of graph $K_m \square C_n$, each column has exactly two missing colors. Similarly, it is not hard to see that the set of vertices $\{ (2 \lceil j/2 \rceil - r, j) \mid 1 \leq j \leq \lceil m/2 \rceil + 1, r = 0, 1 \}$ is a $b$-dominating system. Because for $1 \leq k \leq \lceil m/4 \rceil$, the missing colors of column $2k$ are $4k - 3$ and $4k - 2$, while we have $c(2k, 2k+1) = c(2k, 2k-1) = 4k - 3$ and $c(2k-1, 2k+1) = c(2k-2, 2k-1) = 4k - 2$. Moreover, the missing colors of column $2k - 1$ are $4k - 1$ and $4k$, while we have $c(2k, 2k-2) = c(2k-1, 2k) = 4k - 1$ and $c(2k-1, 2k-2) = c(2k, 2k) = 4k$.

Now assume $4 \leq m \leq 8$ and $m \leq 2n - 2$. In Figure 3 we provide a $b$-coloring of graphs $K_4 \square C_n$, $n = 4, 5$ and $K_7 \square C_n$, $n = 5, 6$. In these colorings the $b$-dominating system, $S$ is the set of circled vertices. Then we apply Lemma 2 for the given coloring of $K_4 \square C_4$ twice, first for $(r, s) = (3, 4)$ and second for $(r, s) = (2, 3)$. Also, we apply that lemma for the given coloring of graph $K_4 \square C_5$, twice, first for $(r, s) = (3, 4)$ and second for $(r, s) = (3, 4)$. Thus we obtain the desired $b$-colorings of graphs $K_m \square C_n$, $m = 5, 6$, $n = 4, 5$. Moreover, we apply Lemma 2 for the given colorings of graphs $K_7 \square C_5$ and $K_7 \square C_6$ for $(r, s) = (6, 5)$ and obtain the desired $b$-colorings of graphs $K_8 \square C_n$, $n = 5, 6$. By Corollary 2, to obtain a $b$-coloring of graph $K_m \square C_n$, $n \geq t$, it is enough to have a $b$-coloring of graphs $K_m \square C_t$ and $K_m \square C_{t+1}$. Therefore, from the $b$-coloring obtained above we have the desired $b$-coloring of graphs $K_m \square C_n$, $4 \leq m \leq 9$ and $m \leq 2n - 2$. □
4 \textit{b}-chromatic number of graph $K_m \square P_n$

In this section, by using the results of Section 2, we determine the exact value of $\varphi(K_m \square P_n)$. We know that $\chi(K_m \square P_n) = m$ and $\Delta(K_m \square P_n) = m + 1$. Therefore by (1),

\[ m \leq \varphi(K_m \square P_n) \leq m + 2. \quad (3) \]

\textbf{Theorem 2.} For positive integers $m, n \geq 4$:

\[ \varphi(K_m \square P_n) = \begin{cases} 
  m & \text{if } m \geq 2n - 2, \\
  m + 1 & \text{if } 2n - 5 \leq m \leq 2n - 3, \\
  m + 2 & \text{if } m \leq 2n - 6.
\end{cases} \]

\textbf{Proof.} Assume $m \geq 2n - 2$. By Corollary 1, $\varphi(K_m \square P_n) \leq 2(n - 1)$. Hence by (3), $\varphi(K_m \square P_n) = m$.

If $\varphi(K_m \square P_n) = m + 2$, then there is not any \textit{b}-dominating vertex in the first and the last columns of graph $K_m \square P_n$, because the vertices in the first and the last columns are of degree $m$. Furthermore, by Proposition 1, the other $n - 2$
columns each contains at most two $b$-dominating vertices. Therefore, $m + 2 = \varphi(K_m \square P_n) \leq 2(n - 2)$. Hence for $m \geq 2n - 5$, we have $\varphi(K_m \square P_n) \leq m + 1$.

Now let $2n - 5 \leq m \leq 2n - 3$, we present a $b$-coloring of graph $K_m \square P_n$ by $m + 1$ colors. We consider two cases.

**Case 1.** $m = 2n - 3$.

We define a coloring $c : V(K_m \square P_n) \rightarrow \{1, 2, \ldots, m + 1\}$ by:

$$c(i, j) = \begin{cases} 
  m - 1 & \text{if } (i, j) = (m, 1), \\
  m + 1 & \text{if } (i, j) = (3j - 4, j), 1 \leq j \leq n - 1, \\
  i + j - 1 \pmod{m} & \text{otherwise.}
\end{cases}$$

It is not hard to see that the above assignment is a proper coloring of graph $K_m \square P_n$. In fact this assignment presents a partial circular latin rectangle with the rest entries filled as above.

The set $S = \{(m - 1, 1), (3n - 5, n), (3j - 5, j), (3j - 3, j) \mid 2 \leq j \leq n - 1\}$ (the summations are modulo $m$) is a $b$-dominating system. Obviously, each vertex dominates $m - 1$ neighbors on its column, which are in different color classes. So for a vertex to be a $b$-dominating vertex it is enough to dominate a vertex with the color which is missed in its column. The missing color in column $j$, $2 \leq j \leq n - 1$ is $4j - 5$, in column 1 is $m$ and in column $n$ is $4n - 7$. Moreover, we have $c(m - 1, 1) = m$, $c(3n - 5, n - 1) = 4n - 7$, $c(3j - 5, j + 1) = 4j - 5$, and $c(3j - 3, j - 1) = 4j - 5$. Therefore, the set $S$ is $b$-dominating system of colors $\{1, 2, \ldots, m + 1\}$. In Figure 5(a), this coloring is shown for $m = 5$, where the circled vertices are $b$-dominating vertices.

Now let $m = 2n - 5$, consider a $b$-coloring of graph $K_m \square P_{n-1}$ by $m + 1$ colors as above. We add a column and color it the same as column 1. This yields a $b$-coloring of graph $K_m \square P_n$ by $m + 1$ colors.

**Case 2.** $m = 2n - 4$.

As illustrated in Figure 4, $\varphi(K_4 \square P_4) = 5$, the $b$-dominating vertices are circled.
Figure 4: A b-coloring of graph $K_4 \square P_4$ by 5 colors.

Assume $n \geq 5$, we define the coloring $c : V(K_m \square P_n) \rightarrow \{1, 2, \ldots, m+1\}$ by:

\[
 c(i, j) = \begin{cases} 
 m - 1 & \text{if } (i, j) = (m, 1), \\
 m + 1 & \text{if } (i, j) = (3j - 4, j), 1 \leq j \leq \lceil \frac{n}{2} \rceil, \\
 m + 1 & \text{if } (i, j) = (3j - 5, j), \lceil \frac{n}{2} \rceil + 1 \leq j \leq n - 1, \\
 m + 1 & \text{if } (i, j) = (3n - 7, n), \\
 i + j - 1 \pmod{m} & \text{otherwise.}
\end{cases}
\]

It is not hard to see that, the assignment above is a proper coloring of graph $K_m \square P_n$. Similar to Case 1, it can be easily checked that the set \{(m − 1, 1), (3n − 6, n), (3j − 5, j), (3j − 3, j), (i, 3i − 6), (i, 3i − 4) | \lceil \frac{n}{2} \rceil + 1 \leq i \leq n - 1, 2 \leq j \leq \lceil \frac{n}{2} \rceil\} (the summations in the first components are modulo $m$ and in the second components are modulo $n$) is a $b$-dominating system. In Figure 5(b) this coloring is shown for $m = 6$, which the circled vertices are $b$-dominating vertices.

Figure 5: A $b$-coloring of graphs $K_5 \square P_4$ and $K_6 \square P_5$ by 6 and 7 colors.

Now assume $m \leq 2n - 6$, and let $n' = n - 2$. Since $m \leq 2n' - 2$, by Theorem 1, $\varphi(K_m \square C_{n'}) = m + 2$, $n' \geq 4$. Hence by Corollary 2, $\varphi(K_m \square P_n) \geq m + 2$. Therefore by (3), $\varphi(K_m \square P_n) = m + 2$, for $n \geq 6$.

For $n = 5$ a $b$-coloring of graph $K_m \square P_n$ is shown in Figure 6, the $b$-dominating vertices are circled. □
**5 \ b\text{-}chromatic number of graph \ K_n \square \ K_n**

We know that $\chi(K_n \square K_n) = n$ and $\Delta(K_n \square K_n) = 2n - 2$. So by (1), $n \leq \varphi(K_n \square K_n) \leq 2n - 1$. In this section we improve these bounds and prove that $2n - 3 \leq \varphi(K_n \square K_n) \leq 2n - 2$. Finally we provide a conjecture that $\varphi(K_n \square K_n) = 2n - 3$, $n \geq 5$.

**Lemma 3.** Let $c$ be a $b$-coloring of graph $K_n \square K_n$ by $2n - 1$ colors. If two vertices $(i, j)$ and $(i, t)$ are $b$-dominating vertices in the $b$-coloring $c$, then in columns $j$ and $t$ there are no other $b$-dominating vertices.

**Proof.** Let $c$ be a $b$-coloring of graph $K_n \square K_n$ by $2n - 1$ colors. It is obvious that if a vertex $(x, y)$ is a $b$-dominating vertex in the $b$-coloring $c$, then all its $2n - 2$ neighbors must have different colors. So the colors of the vertices in the row $x$ and the column $y$ are different. Now, assume to the contrary that the vertices $(i, j)$, $(i, t)$ and $(i', j)$, $i' \neq i$, are $b$-dominating vertices. Since the vertex $(i, t)$ is a $b$-dominating vertex, the vertices in row $i$ and column $t$ all have different colors. Therefore, if $c(i', t) = a$, then no vertex in row $i$ has color $a$. On the other hand the vertex $(i, j)$ is a $b$-dominating vertex, hence in column $j$ we must have a vertex with color $a$. Now, in both row $i'$ and column $j$ we have vertices by color $a$. It contradicts our assumption that the vertex $(i', j)$ is a $b$-dominating vertex. By the same reason the vertex $(i', t)$, for $i' \neq i$, is not $b$-dominating vertex. \qed

**Theorem 3.** For every positive integer $n \geq 2$, we have 

$$\varphi(K_n \square K_n) \leq 2n - 2.$$ 

**Proof.** We know that $\varphi(K_n \square K_n) \leq 2n - 1$. Let $\varphi(K_n \square K_n) = 2n - 1$ and $c$ be a $b$-coloring by $2n - 1$ colors. Without loss of generality we assume that rows 1
to each has at least two \( b \)-dominating vertices and rows \( r + 1 \) to \( n \) each has at most one \( b \)-dominating vertex. Moreover, without loss of generality, we assume that the \( b \)-dominating vertices in the first \( r \) rows are in the first \( s \) columns.

By Lemma 3, in each column \( j \), \( 1 \leq j \leq s \), there is only one \( b \)-dominating vertex. If \( r = 0 \) or \( s = n \), then we have at most \( n \) \( b \)-dominating vertices which is a contradiction. The size of the \( b \)-dominating system in coloring \( c \) is at most \( s + (n-r) \). Now if \( r > 0 \) and \( s < n \), then the number of \( b \)-dominating vertices is at most \( s + (n-r) \leq 2n - 1 - r < 2n - 1 \) which also contradicts our assumption.

\( \square \)

**Theorem 4.** For every positive integer \( n \geq 5 \), we have

\[ \varphi(K_n \Box K_n) \geq 2n - 3. \]

**Proof.** We present a \( b \)-coloring \( c \) by \( 2n - 3 \) colors, for two cases \( n \) odd and \( n \) even. First, we define a function \( f : \mathbb{N} \to \mathbb{Z} \) by:

\[
f(x) = \begin{cases} 
  x & \text{x is odd}, \\
  x - 2 & \text{x is even}.
\end{cases}
\]

**Case 1.** \( n \) is odd.

In this case we define the assignment \( c : V(K_n \Box K_n) \to \mathbb{N} \) by:

\[
c((i, j)) = \begin{cases} 
  i + j - 1 \pmod{n - 1} & i \leq j \leq n - i - 1, \\
  f(i + j) \pmod{n - 1} & n - i \leq j \leq n - 2, i \leq j, \\
  (i + j - 2 \pmod{n - 2}) + (n - 1) & n - 1 < i \leq n - 1, \\
  n - 3 & (i, j) \neq (n - 1, n - 2)
\end{cases}
\]

For columns \( n - 1 \), \( n \) and row \( n \), the assignment \( c \) is as follows.

\[
c((i, n - 1)) = \begin{cases} 
  2i - 2 \pmod{n - 1} & 1 \leq i \leq \frac{n - 1}{2}, \\
  2i - 1 \pmod{n - 1} & \frac{n + 1}{2} \leq i \leq n - 2, \\
  2n - 4 & i = n - 1.
\end{cases}
\]

\[
c((i, n)) = \begin{cases} 
  (2i - 2 \pmod{n - 2}) + (n - 1) & i \text{ odd, } i \leq n - 2, \\
  i - 2 \pmod{n - 1} & i \text{ even, } i \leq n - 2, \\
  n - 2 & i = n - 1.
\end{cases}
\]
c((n, j)) = \begin{align*}
&\begin{cases}
  j - 1 \pmod{n - 1} & j \text{ odd}, j \leq n - 3, \\
  (2j - 2) \pmod{n - 2} + (n - 1) & j \text{ even}, j = n - 2, \\
  2n - 5 & j = n,
\end{cases} \\
&\begin{cases}
  j & j - 1 \pmod{n - 1}, j \leq n - 3 , \\
  2n - 5 & j = n - 2, \\
  1 & j = n.
\end{cases}
\end{align*}

The assignment \( c \) is a \( b \)-coloring and the set \( S = \{(i, i), (j + 1, j) \mid 1 \leq i \leq n - 1, 1 \leq j \leq n - 2\} \) is a \( b \)-dominating system. Because the vertices in \( S \) all have different colors and for each vertex in \( S \) the colors in its row and columns all have different colors except two entries. As an example such a coloring for \( n = 7 \) is illustrated in Figure 7, the \( b \)-dominating vertices are circled.

Case 2. \( n \) is even.

In this case we define the assignment \( c : V(K_n \Box K_n) \to \mathbb{N} \) by:

\[
\begin{align*}
c((i, j)) = \begin{cases}
i + j - 2 \pmod{n - 2} & i + 1 \leq j \leq n - i - 1, \\
f(i + j - 1) \pmod{n - 2} & n - i \leq j \leq n - 2, i + 1 \leq j, \\
(i + j - 1) \pmod{n - 1} + (n - 2) & j \leq i \leq n - 1, \\
i + 1 \neq (n - 1, n - 2) & (i, j) \neq (n - 1, n - 2), \\
i + 1 = (n - 1, n - 2) & (i, j) = (n - 1, n - 2).
\end{cases}
\end{align*}
\]

For columns \( n - 1, n \) and row \( n \), the assignment \( c \) is as follows.

\[
c((i, n - 1)) = \begin{cases}
2i - 2 \pmod{n - 2} & 1 \leq i \leq \frac{n - 2}{2}, \\
2i - 1 \pmod{n - 2} & \frac{n}{2} \leq i \leq n - 3, \\
2n - 5 & i = n - 2, \\
2n - 4 & i = n - 1, \\
n - 3 & i = n.
\end{cases}
\]
\[
c(i, n) = \begin{cases} 
(2i \mod (n-1)) + (n-2) & i \text{ odd, } i \leq n-2, \\
i - 2 \mod (n-2) & i \text{ even, } i \leq n-2, \\
n - 3 & i = n - 1, \\
1 & i = n.
\end{cases}
\]

\[
c(n, j) = \begin{cases} 
(2j - 2 \mod (n-1)) + (n-2) & j \text{ odd, } 3 \leq j \leq n-3, \\
j - 2 \mod (n-2) & j \text{ even, } 3 \leq j \leq n-3, \\
n - 4 & j = 1, \\
2n - 5 & j = n - 2.
\end{cases}
\]

The assignment \( c \) is a \( b \)-coloring and the set \( S = \{(i, i), (j - 1, j) \mid 1 \leq i \leq n - 1, 2 \leq j \leq n - 2\} \cup \{(n - 1, n - 2)\} \) is \( b \)-dominating system. Because the vertices in \( S \) all have different colors and for each vertex in \( S \) the colors in its row and columns all have different colors except two entries. As an example such a coloring for \( n = 8 \) is illustrated in Figure 8, the \( b \)-dominating vertices are circled.

\[\Box\]

\begin{center}
\begin{tabular}{cccccccc}
(7) & (1) & 2 & 3 & 4 & 5 & 6 & 8 \\
8 & (9) & (3) & 4 & 5 & 1 & 2 & 6 \\
9 & 10 & (11) & (5) & 1 & 6 & 4 & 12 \\
10 & 11 & 12 & (13) & (6) & 3 & 1 & 2 \\
11 & 12 & 13 & 7 & (8) & (2) & 3 & 9 \\
12 & 13 & 7 & 8 & 9 & (10) & 11 & 4 \\
13 & 7 & 8 & 9 & 10 & (4) & (12) & 5 \\
4 & 6 & 10 & 2 & 7 & 11 & 5 & 1 \\
\end{tabular}
\end{center}

Figure 8: A \( b \)-coloring of graphs \( K_8 \Box K_8 \) by 13 colors.

Remark. For \( n = 3 \) the only way to have a \( b \)-coloring by 4 colors is Figure 9(a), with the circled vertices as \( b \)-dominating vertices; which is impossible, so \( \varphi(K_3 \Box K_3) = 3 \). For \( n = 4 \) there is a \( b \)-coloring of graph \( K_4 \Box K_4 \) by \( 2n - 2 = 6 \) colors, see Figure 9(b).

Finally, we propose the following conjecture.

Conjecture 1. For every positive integer \( n \geq 5 \), \( \varphi(K_n \Box K_n) = 2n - 3 \).
Figure 9: A partial $b$-coloring of graphs $K_3 \Box K_3$ and $K_4 \Box K_4$.

References

