Proof: Let \((u, v) \in E(G)\). Since \(g(G) \geq 4\) we have \(N(u) \cap N(v) = \emptyset\). Then there are at most \(2k(k-1) + 2\) vertices at distance at most three from at least one of the vertices \(u\) and \(v\). Since \(e(G) = \frac{n^2}{2}\) it follows that there are at least \(\frac{n^2}{2} \cdot \frac{\kappa}{\theta} - \frac{n^2}{2}\) edges whose neighbors are disjoint (since \(g(G) \geq 4\)). Label \(st(u) = st(v) = 1\). Then \(2k\) vertices (including \(u\) and \(v\)) are dominated. Hence, running over all such edges we have that at least \(\frac{n^2}{2} \cdot \frac{\kappa}{\theta} - \frac{n^2}{2}\) \(\cdot 2k\) vertices are dominated. The rest of the vertices are labeled 0.

Hence,

\[
\gamma_s(G; 0, 1) \leq n - k \cdot \frac{n^2}{2} \cdot \frac{\kappa}{\theta} - \frac{n^2}{2} + \frac{n^2}{2} \cdot \frac{\kappa}{\theta} = n - n \cdot \frac{k}{\theta} = n \left(1 - \frac{k}{\theta}\right) \leq n \left(1 - \frac{1}{2}\right).
\]

References


Chromatic Equivalence Classes of Certain Generalized Polygon Trees, II

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ABSTRACT

Let \(P(G)\) denote the chromatic polynomial of a graph \(G\). Two graphs \(G\) and \(H\) are chromatically equivalent, written \(G \sim H\), if \(P(G) = P(H)\). A graph \(G\) is chromatically unique if for any graph \(H, G \sim H\) implies that \(G\) is isomorphic with \(H\). In "Chromatic Equivalence Classes of Certain Generalized Polygon Trees", Discrete Mathematics Vol.172, 103 - 114 (1997), Peng et al. studied the chromaticity of certain generalized polygon trees. In this paper, we present a chromaticity characterization of another big family of such graphs.

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1. Introduction

The graphs that we consider are finite, undirected and simple. Let \(P(G)\) denote the chromatic polynomial of a graph \(G\). Two graphs \(G\) and \(H\) are said to be chromatically equivalent, and we write \(G \sim H\), if \(P(G) = P(H)\). A graph \(G\) is chromatically unique if \(G \sim H\) implies that \(H\) is isomorphic to \(G\). A set of graphs \(S\) is called a chromatic equivalence class if any two element of \(S\) are

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chromatically equivalent, and if any graph which is chromatically equivalent with a graph $G$ in $S$ is also isomorphic to some element of $S$. Although chromatically unique graphs have been the subject of many recent papers (see [2] and [3]), relatively few results concerning the chromatic equivalence classes of graphs are known.

![Figure 1. $G_t^a(a, b; c, d)$](image)

A path in $G$ is called simple if the degree of each interior vertex is two in $G$. A generalized polygon tree is a graph defined recursively as follows. A cycle $C_p$ $(p \geq 3)$ is a generalized polygon tree. Next, suppose $H$ is a generalized polygon tree containing a simple path $P_k$, where $k \geq 1$. If $G$ is a graph obtained from the union of $H$ and a cycle $C_r$, where $r > k$, by identifying $P_k$ in $H$ with a path of length $k$ in $C_r$, then $G$ is also a generalized polygon tree. Consider the generalized polygon tree $G_t^a(a, b; c, d)$ shown in Figure 1. The integers $a, b, c, d, s$ and $t$ represent the lengths of the respective paths between the vertices of degree $> 2$, where $s \geq 0, t \geq 0$. Without loss of generality, assume that $a \leq b, a \leq c \leq d$ and if $a = c$, then $b \leq d$. Thus, $\min\{a, b, c, d\} = a$. Let $r = s + t$. We now form a family $C_r(a, b; c, d)$ of the graphs $G_t^a(a, b; c, d)$ where the values of $a, b, c, d$ and $r$ are fixed but the values of $s$ and $t$ vary; that is

$$C_r(a, b; c, d) = \{ G_t^a(a, b; c, d) | r = s + t, s \geq 0, t \geq 0 \}.$$ 

It is clear that the families $C_0(a, b; c, d)$ and $C_1(a, b; c, d)$ are singletons.

Note that $G_t^a(a, b; c, d)$ is a connected $(n, n + 2)$-graph, whose chromatic polynomials were computed by Chao and Zhao (see [1]), who also determined several chromatic equivalence classes, excluding among others the graph $G_t^a(a, b; c, d)$.

In [6], Peng et al. showed that $C_r(a, b; c, d)$ is a chromatic equivalence class for $a, b, c, d$ at least $r + 3$. As a corollary, the graph $G_t^a(a, b; c, d)$ is chromatically unique for $a, b, c, d$ at least four (see also Peng [5]). In [4], Oromoi and Peng characterized the chromaticity of $C_r(a, b; c, d)$ for the minimum $a, b, c, d$ and $r$ less than four. In this paper, we characterize the chromaticity of $C_r(a, b; c, d)$ for the minimum $a, b, c$, and $d$ equal to $r + 2$, and $r \geq 2$. Also we discover that the following conjecture is not true for each $r \geq 2$.

Conjecture [6]. The family of graphs $C_r(a, b; c, d)$ is a chromatic equivalence class whenever $a, b, c$, and $d$ are each at least four.

In the remaining of this section, we give some known results that will be used to prove our main theorem. The girth of $G$, denoted by $g(G)$, is the length of a shortest cycle of $G$.

Theorem A (Whitney [7]). Let $G$ and $H$ be chromatically equivalent graphs. Then

(a) $|V(G)| = |V(H)|$;

(b) $|E(G)| = |E(H)|$;

(c) $g(G) = g(H)$;

(d) $G$ and $H$ have the same number of shortest cycles.

Theorem B (Chao and Zhao [1], Peng et al. [6]). All the graphs in $C_r(a, b; c, d)$ are chromatically equivalent.

By this theorem we only need to compute $P(G_t^a(a, b; c, d))$ for computing the chromatic polynomial of $G_t^a(a, b; c, d)$.

The next result is Case 1 in the proof of Theorem 6 in [6].

Theorem C (Peng et al. [6]). If $G_t^a(a, b; c, d)$ and $G_t^a(a', b; c', d)$ are chromatically equivalent and $s + t = s' + t'$, then $G_t^a(a, b; c, d) \in C_r(a, b; c, d)$, where $r = s + t$.

The next result gives the chromatic polynomial of $G_t^a(a, b; c, d)$. In [1], Chao and Zhao also determined the chromatic polynomial of this graph, but we shall use the computed chromatic polynomial of $G_t^a(a, b; c, d)$ in [6] to prove our main results.

Theorem D (Peng et al. [6]). Let the order of $G_t^a(a, b; c, d)$ be $n = a + b + c + d + r - 2$, and $z = 1 - \lambda$. Then we have

$$P(G_t^a(a, b; c, d)) = \frac{(-1)^n \lambda}{(z - 1)^2} \cdot Q(G_t^a(a, b; c, d)).$$
where
\[
Q(G^r_t(a, b; c, d)) = (x^{n+1} - x^{n+e} - x^{n+e+1} - x^{n+e+2} + x^{n+e} - x) - \\
(1 + x + x^2) + (x + 1)(z^r + z^{r+1} + z^{r+2}) - \\
(z^{n+c} + x^{n+e} + x^{n+c+e} + x^{n+c+e+1}).
\]

2. Main Theorem

In this section, we shall characterize the chromaticity of the family \(C_r(a, b; c, d)\) when \(\min{a, b, c, d} = r + 2\), which gives us two counterexamples for the conjecture.

**Theorem 1.** The family of graphs in \(C_r(a, b; c, d)\) is a chromatic equivalence class if \(r \geq 2\) and \(\min{a, b, c, d} = r + 2\), except the two families \(C_r(a, b; b + 1, b + r + 2)\) and \(C_r(r + 2, c + r + 2; c, c + 1)\).

**Proof.** Let \(G = G^r_t(a, b; c, d) \in G_r(a, b; c, d)\) and \(H \sim G^r_t(a, b; c, d)\). By Lemma 4 and Theorem 2 in [I], \(H \cong G^r_t(a', b'; c', d')\), where \(a', b', c', d' \geq 1\). Let \(r' = r + t\).

If \(r' = r\), then by Theorem C., \(H \in C_r(a, b; c, d)\). Now assume \(r' \neq r\). We solve the equation \(Q(G^r_t) = Q(H)\). After cancelling the terms \(x^{n+1}, -x\) and \((-1 + x + x^2)\), we have \(Q_1(G) = Q_1(H)\) where
\[
Q_1(G) = z^{r+1} + (z + 1)(x^r + x^{r+1} + x^{r+2}) = z^{n+e} - x^{n+e} - x^{n+e+1} - x^{n+e+2} + x^{n+e} - x - \\
(1 + x + x^2) + (x + 1)(z^r - z^{r+1} - z^{r+2} - z^{r+3} + z^{r+4} - z^{r+5}),
\]
\[
Q_1(H) = z^{r+1} + (z + 1)(x^r + x^{r+1} + x^{r+2}) - z^{n+e} - x^{n+e} - x^{n+e+1} - x^{n+e+2} - x^{n+e} - x - \\
(1 + x + x^2) + (x + 1)(z^r - z^{r+1} - z^{r+2} - z^{r+3} + z^{r+4} - z^{r+5}),
\]
and \(a + b + c + d + r = a' + b' + c' + d' + r; a \leq b, a \leq c \leq d; a' \leq b', a' \leq c' \leq d'\).

Since by assumption \(\min{a, b, c, d} = r + 2\), the term \(z^{n+1}\) in \(Q_1(G)\) cannot be cancelled. Hence \(z^{n+1}\) is in \(Q_1(H)\) and this implies \(r = \min\{r + 1, a', b', c', d'\}\). Thus \(r + 1 = r + 1\) or \(a' = r + 1\). By our assumption, we must have \(a' = r + 1\). Since \(a = \min\{a, b, c, d\} = r + 2\), we have \(Q_2(G) = Q_2(H)\) where
\[
Q_2(G) = z^{r+2} + (z + 1)(x^r + x^{r+1} + x^{r+2}) - z^{r+2} - x^{r+2} - x^{r+3} - x^{r+4} - x^{r+5},
\]
\[
Q_2(H) = z^{r+2} + (z + 1)(x^r + x^{r+1} + x^{r+2}) - z^{r+2} - x^{r+2} - x^{r+3} - x^{r+4} - x^{r+5},
\]
and \(b + c + d + r + 1 = b' + c' + d' + r; r + 2 = a, r + 2 \leq c \leq d, a' + 1 \leq b', r + 1 \leq c' \leq d'\).

The lowest power positive term in \(Q_2(G)\) cannot be cancelled, hence we have \(\min\{b', c', d', r' + 1\} = r + 2\). The term \(z^{r+1}\) in \(Q_2(G)\) cannot be cancelled. Hence \(z^{r+1}\) is a term in \(Q_2(H)\), and thus we have \(r' + 1 = r + 3\) or \(b' = r + 3\) or \(c' = r + 3\) or \(d' = r + 3\) or \(r + 1 = r + 3\) or \(b' + 1 = r + 3\). Since \(b, c, d \geq r + 1\), \(b' = r + 3\) because \(a' = r + 1\). Thus \(b' = r + 2\) is impossible. Also \(d' = r + 3\) or \(r' + 1 = r + 3\).

Therefore we need to consider only the first four cases (underline).

**Case 1.** Suppose \(r' + 1 = r + 3\) or \(r' = r + 2\). Then we have \(Q_3(G) = Q_3(H)\) where
\[
Q_3(G) = (z + 1)(x^r + x^{r+1} + x^{r+2}) - z^{r+2} - x^{r+2} - x^{r+3} - x^{r+4} - x^{r+5},
\]
\[
Q_3(H) = (z + 1)(x^r + x^{r+1} + x^{r+2}) - z^{r+2} - x^{r+2} - x^{r+3} - x^{r+4} - x^{r+5},
\]
and \(b + c + d = b' + c' + d' + 1; r + 2 \leq b, r + 2 \leq c \leq d, r + 3 \leq b', r + 2 \leq c' \leq d'\).

It is easy to see that \(\min\{b, c, d\} = \min\{b', c', d'\}\). We consider two subcases: \(b \leq c\) and \(b > c\).

**Subcase 1.1.** Suppose \(b \leq c\). Then we have \(\min\{b, c, d\} = b\) and \(g(G) = a + b\).

Also we have \(b = b'\) (if \(b' \leq c'\)) or \(b = c'\) (if \(b' > c'\)). If \(b' = b\), then \(g(G) = a' + b' = g(G) = a + b\), and we have \(a = a', a\), a contradiction (since \(a = r + 2\) and \(a' = r + 1\)). Hence we have \(b = c'\) and \(g(G) = a + b = a + c' = r + 2 + c'\).

Then \(g(H)\) is equal to either \(a' + b'\) or \(c' + d'\) or \(a' + r' + c' + d' = 2r + 3\) or \(c' + d'\). Since \(g(H) = g(G)\), the last possibility is impossible. We now look at the other two possibilities.

**Subcase 1.1.1** Suppose \(g(G) = a' + b' = r + 1 + b'\) (or \(c'\)). Then \(g(G) = r + 2 + c' = r + 1 + b'\) and we have \(b' = c + b + 1\). Moreover, we have \(Q_4(G) = Q_4(H)\) where
\[
Q_4(G) = (z + 1)(x^r + x^{r+1} + x^{r+2}) - z^{r+2} - x^{r+2} - x^{r+3} - x^{r+4} - x^{r+5},
\]
\[
Q_4(H) = (z + 1)(x^r + x^{r+1} + x^{r+2}) - z^{r+2} - x^{r+2} - x^{r+3} - x^{r+4} - x^{r+5},
\]
and \(c + d = b' + c' + 1; r + 2 \leq b, c \leq d, r + 3 \leq b', r + 1 \leq c' \leq d'\).
It can be seen that the term $z^c$ in $Q_4(G)$ cannot be cancelled since if $c = 2r + b + 2$, then at least one of the terms $z^d$ and $z^{d+1}$ is neither cancelled in $Q_4(H)$ nor is a term of $Q_4(G)$. Thus we must have $z^c$ in $Q_4(H)$. So we have $c = b' + c' = d'$. Suppose $c = b'$ or $c = d'$. Then we have $d = d' + 1$ and from $Q_4(G) = Q_4(H)$, after cancelling equal terms, we have $Q_4(G) = Q_4(H)$ where

$$Q_4(G) = x^{d+1} - x^{r+1+c} - x^{d+r+c} - x^{r+c} - x^{d+r+1} - x^{d+r+2};$$
$$Q_4(H) = x^{d+1} - x^{r+c} - x^{d+c} - x^{r+d+1} - x^{r+1} - x^{d+1}.$$}

The terms $x^{d+1}$ and $x^{d+1}$ must be cancelled in $Q_4(G)$ and $Q_4(H)$, respectively; otherwise, there is no solution. The term $x^{d+1}$ can be cancelled with $x^{r+b+2}$ or $x^{r+b+2}$, and the term $x^{d+1}$ can be cancelled with $x^{r+b+4}$ or $x^{r+b+4}$. If $d + 1 = 2r + b + 2$ then $d - 1 = 2r + b$ and $Q(4) = Q(H)$ has no solution. If $d + 1 = r + c + 1$, then $d - 1 = r + c = r + b + 1$ and we have many solutions: $a = r + 2, c = b + 1, d = b + r + 2$ and $d' = r + 1, b = b + 1, c' = b, d' = b + r + 1$, and $d' = r + 2$. In other words, we have $G_3^0(r+2,b;b+1,b+r+2) = G_3^0(r+1,b+1,b; b+1,r+1)$, but $G_3^0(r+1,b+1,b; b+1,r+1)$ $\notin G_3^0(r+2,b;b+1,b+r+2)$. Note that $G_3^0(r+2,b;b+1,b+r+2) \notin G_3^0(r+1,b+1,b; b+1,r+1)$ where $d' = r + 2$.

Subcase 1.1.1.2 Suppose $c = d'$. Recall that we also have $b = c' = b' - 1$. Then $d = b' + b + 2$ and from $Q_4(G) = Q_4(H)$, after cancelling equal terms, we have $Q_4(G) = Q_4(H)$ where

$$Q_4(G) = x^{d+1} - x^{r+b+2} - x^{r+c} - x^{d+c} - x^{d+r+1} - x^{d+r+2};$$
$$Q_4(H) = x^{d+1} - x^{r+c} - x^{d+c} - x^{r+d+1} - x^{r+1} - x^{d+1};$$

Now $x^{d+1}$ in $Q_4(H)$ cannot be cancelled because $b \leq c$ but $x^{d+1}$ is not a term in $Q_4(G)$. This is a contradiction.

Subcase 1.1.2 Suppose $g(H) = c' + d'$. Then $g(G) = r + 2 + c' = c' + d'$, thus $d' = r + 2$. Since $r + 2 \leq b = c' \leq d' = r + 2$, we have $c' = r + 2 = b$. From $Q_4(G) = Q_4(H)$, we have $Q_4(G) = Q_4(H)$ where

$$Q_4(G) = (x + 1)(x^d + x^c) - x^{r+b+1};$$
$$Q_4(H) = (x + 1)(x^d + x^c) - x^{r+b+1};$$
$$Q_4(G) = x^{r+b+1} - x^{r+b+2} - x^{r+b+1} - x^{r+b+1};$$
$$Q_4(H) = x^{r+b+1} - x^{r+b+2} - x^{r+b+1} - x^{r+b+1};$$

and $b + d = b' + d' + 1, a = r + 2 \leq b, c \leq d, c = b; a = r + 1 \leq b', c' \leq d', c' = b, b \leq c, b' \geq r + 3$. Note that there is at least one positive term in $Q_4(G)$ that cannot be cancelled by a negative term. This can be seen as follows. For the case of $b < d$, at least one of the terms $x^b$ or $x^{d+1}$ cannot be cancelled. Also for the case of $b \geq d$, at least one of the terms $x^b$ or $x^{d+1}$ cannot be cancelled. Now consider the positive terms in $Q_4(G)$ and $Q_4(H)$. Since $Q_4(G) = Q_4(H)$, we have eight possibilities: $b = b', b = b' + 1, b = d', b = d' + 1, b + 1 = b', b + 1 = d', d + 1 = b', d + 1 = d'$.}

Subcase 1.2.1 Suppose $b = b'$. Then $d = d' + 1$, and from $Q_4(G) = Q_4(H)$, after cancelling equal terms, we have $Q_4(G) = Q_4(H)$ where

$$Q_4(G) = x^{d+1} - x^{r+b+2} - x^{r+c} - x^{d+c} - x^{d+r+1} - x^{d+r+2};$$
$$Q_4(H) = x^{d+1} - x^{r+c} - x^{d+c} - x^{r+d+1} - x^{r+1} - x^{d+1};$$

Note that $x^{d+1}$ and $x^{d+1}$ must be cancelled in $Q_4(G)$ and $Q_4(H)$, respectively, but this is impossible. Therefore there is no solution.
\[ Q_{10}(G) = x^{d+1} - x^{2r+4} - x^{r+e+2} - x^{r+e} - x^{e+d+2} - x^{d+1}, \]
\[ Q_{10}(H) = x^{d+1} - x^{2r+4} - x^{r+e+1} - x^{r+e} - x^{e+d+2} - x^{d+1}. \]

The term \(-x^{r+e+1}\) is in \(Q_{10}(H)\), but \(-x^{r+e+2}\) is not in \(Q_{10}(G)\) (since \(c < b\) and \(c \leq d\)). Therefore it must be cancelled by a positive term in \(Q_{10}(H)\). Thus \(r + c + 1 = d - 1\), hence \(d + 1 = r + c + 3\) and the term \(x^{d+1}\) cannot be cancelled in \(Q_{10}(G)\), which is a contradiction.

**Subcase 1.2.2.4** Suppose \(b = d' + 1\). Then \(d = b\), and from \(Q_5(G) = Q_5(H)\), after cancelling equal terms, we have \(Q_{11}(G) = Q_{11}(H)\) where
\[ Q_{11}(G) = x^{d+1} - x^{2r+4} - x^{r+e+2} - x^{r+e} - x^{e+d+2} - x^{d+2}, \]
\[ Q_{11}(H) = x^{d+1} - x^{2r+4} - x^{r+e+1} - x^{r+e} - x^{e+d+2} - x^{d+2}. \]

The term \(-x^{r+e+1}\) is in \(Q_{10}(H)\), but \(-x^{r+e+2}\) is not in \(Q_{10}(G)\) (since \(c < b\) and \(c \leq d\)). Therefore it must be cancelled by a positive term in \(Q_{10}(H)\). Thus \(r + c + 1 = b - 1\), hence \(b + 1 = r + c + 3\) and the term \(x^{d+2}\) can be cancelled in \(Q_{11}(G)\), only if \(b + 1 = r + c + 3 = r + d + 2\) or \(d = c + 1\). Thus the term \(-x^{r+e+2}\) cannot be cancelled in \(Q_{11}(G)\), and \(-x^{r+e+2}\) is not a term of \(Q_{11}(H)\), which is a contradiction.

**Subcase 1.2.2.5** Suppose \(b + 1 = b'\). Then \(d = d' + 2\), and from \(Q_5(G) = Q_5(H)\), after cancelling equal terms, we have \(Q_{12}(G) = Q_{12}(H)\) where
\[ Q_{12}(G) = x^{d+1} + (x + 1)x^{d - x^{2r+4} - x^{r+e+2} - x^{r+e} - x^{e+d+2} - x^{d+2}}, \]
\[ Q_{12}(H) = x^{d+1} + (x + 1)x^{d - x^{2r+4} - x^{r+e+1} - x^{r+e} - x^{e+d+2} - x^{d+2}}. \]

The term \(-x^{r+e+1}\) is in \(Q_{12}(H)\), but it is not in \(Q_{12}(G)\) (since \(c < b\) and \(c \leq d\)). Hence \(-x^{r+e+1}\) must be cancelled by a positive term in \(Q_{12}(H)\). We have either \(r + c + 1 = b + 2\) or \(r + c + 1 = b - 2\) or \(b + 1 = r + c + 3\). If \(r + c + 1 = b + 2\), then the term \(x^{d+1} = x^{r+e+1}\) cannot be cancelled in \(Q_{12}(G)\) which means that \(x^d\) is a term of \(Q_{12}(H)\). Thus we have either \(b = d - 2\) or \(b = d - 1\). In each case the term \(x^d\) cannot be cancelled in \(Q_{12}(G)\), which is a contradiction.

If \(r + c + 1 = b - 2\), then we have the term \(x^{d+2} = x^{r+e+2}\) in \(Q_{12}(H)\) and this term cannot be cancelled in \(Q_{12}(H)\) because \(r + 2 \leq c < b\). Since \(-x^{r+e+2}\) occurs in \(Q_{12}(G)\), we must have the term 2\(x^{r+e+2}\) in \(Q_{12}(G)\), which is impossible.

If \(r + c + 1 = d - 1\), then \(d - 2 = r + c\) and the term \(x^{d-2} = x^{r+e}\) cannot be cancelled in \(Q_{12}(H)\). Thus, we must have the term \(x^{d-2} = x^{r+e}\) in \(Q_{12}(G)\).

This implies \(b = r + c\). Now the term \(x^{d+2} = x^{r+e+2}\) cannot be cancelled in \(Q_{12}(H)\) because \(r + 2 \leq c < b\). Since \(-x^{r+e+2}\) occurs in \(Q_{12}(G)\), we must have the term 2\(x^{r+e+2}\) in \(Q_{12}(G)\), which is impossible. Therefore there is no solution.

In the remaining three possibilities, that is \(b + 1 = d'\), \(d + 1 = b'\), and \(d + 1 = d'\), there is no solution for \(Q(G) = Q(H)\). The proof is similar to that of Case 1. The detailed proof can be obtained by e-mail from the second author or viewed at http://www.fssas.upm.edu.my/yhpen7/pubs/1234.pdf.
Corollary. We discover that the conjecture in [6] is only true for \( r = 1 \). For each \( r \geq 2 \), we provide two counterexamples as follows:

- \( G_r^2(r+2, b; b+b+r+2) \sim G_{r+2}^2(r+1, b+1; b+b+r+1) \) for \( b \geq 4 \) but
  \( G_{r+2}^2(r+1, b+1; b+b+r+1) \not\in \mathcal{G}_r(r+2, b; b+b+r+2) \).
- \( G_r^2(r+2, c+r+2; c+c+1) \sim G_{r+2}^2(r+1, c+r+1; c+c+1) \) for \( c \geq 4 \) but
  \( G_{r+2}^2(r+1, c+r+1; c+c+1) \not\in \mathcal{G}_r(r+2, c+r+2; c+c+1) \).

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References


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A few identities involving partitions with a fixed number of parts

Jean-Lou De Carufel

1 Notation and elementary identities

The partition function \( P(n) \) gives the number of ways of writing an integer \( n \) as a sum of positive integers without regard to the order. Let \( p(n, k) \) be the number of solutions of the diophantine equation

\[
x_1 + x_2 + \ldots + x_k = n,
\]

where \( 0 < x_1 \leq x_2 \leq \ldots \leq x_k \) and denote by \( p_r(n, k) \) the number of solutions of (1) with \( r \leq x_1 \leq x_2 \leq \ldots \leq x_k \).

By solving the recurrence relationship

\[
p(n, m) - p(n - m, m) = p(n - 1, m - 1)
\]

for small values of \( m \), Colman [1] obtained formulas for \( p(n, 1) \), \( p(n, 2) \), \( p(n, 3) \), \( p(n, 4) \), \( p(n, 5) \) and \( p(n, 6) \). With simple combinatorial considerations together with the formulas

\[
\left\lfloor \frac{n-1}{k} \right\rfloor + \left\lfloor \frac{n-2}{k} \right\rfloor + \ldots + \left\lfloor \frac{n-k}{k} \right\rfloor = n-k,
\]

\[
\sum_{i=1}^{3} \left\lfloor \frac{(n-i)^2}{12} \right\rfloor = \begin{cases} \frac{(n-2)^2}{2} & \text{if } n \text{ is even,} \\ \frac{(n-3)(n-1)}{4} & \text{if } n \text{ is odd,} \end{cases}
\]

where \( \lfloor x \rfloor = [x + 1/2] \) is the integer closest to \( x \) (and \( \lfloor \cdot \rfloor \) is the greatest integer function), and the formulæ for the sum of the \( k \)th powers of the first \( n \) integers, we will derive some slick expressions for \( p(n, 1) \), \( p(n, 2) \), \( p(n, 3) \) and \( p(n, 4) \). Finally, we are going to use these identities together with a technique of Hirschhorn (see [2]) to give a formula for \( p(n, 5) \). Our approach and our method appear to be elementary and original and lead to formulæ equivalent to those of Colman [1].

First, let us remark that

\[
p(n, 1) = 1, \quad p(n, 2) = \left\lfloor \frac{n}{2} \right\rfloor, \quad p_r(n, 2) = \left\lfloor \frac{n}{2} \right\rfloor - (r - 1)
\]